Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Geometry and Physics 51 (2004) 244–255
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

# On Einstein four-manifolds

Ézio de Araujo Costa

*Instituto de Matematica, Universidade Federal da Bahia, Av. Ademar de Barros, Ondina, CEP 40170-110 Salvador-Bahia, Brazil*

Received 18 May 2003; received in revised form 21 October 2003

**Abstract**

In this paper we obtain obstructions to the existence of Einstein metrics satisfying auxiliary sectional curvature bounds. In particular, we give sufficient conditions for a compact-oriented Einstein four-manifold $M$ to be isometric to either the sphere $S^4$ or the complex projective space $CP^2$. Also, we improve the Hitchin–Thorpe's inequality which relates the Euler characteristic of $M$ and its signature.


*MSC:* 53C25; 53C24

*JGP SC:* Differential geometry; General relativity

*Keywords:* Four-manifold; Einstein manifold; Sectional curvature; Euler characteristic

## 1. Introduction

A basic problem in Riemannian geometry is to decide if a Riemannian manifold carries an Einstein metric. In particular, in dimension four, the spheres $S^4$, the product of two spheres of same curvature $S^2 \times S^2$, the real projective space $RP^4$ and the complex projective space $CP^2$ are examples of compact Einstein manifolds. Initially we state our results. For this, let $M$ be a compact-oriented four-dimensional Riemannian manifold. The Weyl tensor $W$ of $M$ has a decomposition $W = W^+ \oplus W^-$, where $W^\pm$ are the *self-dual* and *anti-self-dual* Weyl tensors of $M$, respectively. If $M$ is an Einstein four-manifold with Ricci curvature $\rho$ and volume form $dV$, then the Euler characteristic of $M$ and its signature $\tau$ are given by

$$\chi(M) = \frac{1}{8\pi^2}\int_M \left(|W^+|^2 + |W^-|^2 + \frac{2\rho^2}{3}\right) dV \tag{1.1}$$

*E-mail address:* ezio@ufba.br (É. de Araujo Costa).

0393-0440/$ – see front matter © 2003 Elsevier B.V. All rights reserved.
doi:10.1016/j.geomphys.2003.10.013

and

$$\tau(M) = \frac{1}{12\pi^2}\int_M (|W^+|^2 - |W^-|^2)\,\mathrm{d}V. \qquad (1.2)$$

For these Einstein manifolds, holds the classical Hitchin–Thorpe inequality (see [8]).

**Theorem A** (Hitchin). *Let M be a compact-oriented Einstein four-manifold. Then*

$$\chi \geq \tfrac{3}{2}|\tau|. \qquad (1.3)$$

*Moreover, if M has non-negative (non-positive) sectional curvature, then*

$$\chi \geq (\tfrac{3}{2})^{3/2}|\tau|. \qquad (1.4)$$

In [7, Theorem B], Gursky and LeBrun improved the inequality (1.4) (see also [10], Proposition 10.2, p. 281 and Remark 10.1, p. 282).

**Theorem B** (Gursky–LeBrun). *Let M be a compact and oriented Einstein four-manifold with sectional curvature K.*

(i) *If* $K \geq 0$ *and* $W^{\pm} \not\equiv 0$, *then*

$$\chi > \tfrac{15}{4}|\tau|. \qquad (1.5)$$

(ii) *If* $K \leq 0$, *then*

$$\chi \geq \tfrac{15}{8}|\tau|. \qquad (1.6)$$

Our first result is similar to Theorem B (ii).

**Theorem 1.1.** *Let M be a compact-oriented Einstein four-manifold with sectional curvature K and Ricci curvature* $\rho$. *If* $\rho < 0$ *and* $\inf K \geq 2\rho/3$, *then*

$$\chi \geq \tfrac{15}{8}|\tau|. \qquad (1.7)$$

**Example 1.1.** Let $M$ be a oriented four-manifold such that $M$ is homeomorphic to either $\mathrm{CP}^2\sharp j\overline{\mathrm{CP}^2}$, $6 \leq j \leq 8$, $\mathrm{CP}^2\sharp\mathrm{CP}^2\sharp\mathrm{CP}^2$ or $(S^2\times S^2)\sharp j\overline{\mathrm{CP}^2}$, $5 \leq j \leq 8$, where $\sharp$ is the connected sum. By Theorem 1.1, $M$ do not admit an Einstein metric with Ricci curvature $\rho < 0$ and sectional curvature $K$ such that $\inf K \geq 2\rho/3$.

There are few known examples of compact Einstein four-manifolds $M$ with sectional curvature $K \geq 0$. In particular, Berger [2, Theorem 2 and Remark 2] proved that *if there exists* $K_0 > 0$ *such that* $K_0/4 \leq K \leq K_0$, *then M is isometric to either* $S^4$, $\mathrm{RP}^4$ *or* $\mathrm{CP}^2$. In [12, Theorem 1.1a], Yang proved the following: *Let M be a compact Einstein four-manifold with Ricci curvature 1. If* $K \geq (\sqrt{1249} - 23)/20 \simeq 0.102843$, *then M is isometric to either* $S^4$, $\mathrm{RP}^4$ *or* $\mathrm{CP}^2$. A compact-oriented four-manifold $M$ is *definite* if the space of negative (positive) harmonic two-forms $H^2_-(M) = 0\,(H^+_2(M) = 0)$. Gursky and LeBrun [7, Theorem A] proved that *if M is a compact-oriented four-manifold with* $K \geq 0$ *and*

$H^2_+(M) \neq 0$, *then M is isometric to* $\mathrm{CP}^2$. Our next theorem improves the results of Berger, Yang and Gursky/LeBrun.

**Theorem 1.2.** *Let M be a compact Einstein four-manifold with Ricci curvature* $\rho > 0$ *and sectional curvature K. Then we have*

(a) *If* $\sup K \leq 2\rho/3$, *then M is isometric to either* $S^4$, $\mathrm{RP}^4$ or $\mathrm{CP}^2$.
(b) *If* $\rho = 1$ *and* $\inf K \geq (2-\sqrt{2})/6 \simeq 0.097631$, *then M is isometric to either* $S^4$, $\mathrm{RP}^4$ *or* $\mathrm{CP}^2$, *with their normalized metrics*.
(c) *Let* $\rho = 1$, $K \geq 0$ *and M oriented. If* $|W^+|^2 \leq 2/3$, *then M is isometric to either* $S^4$ *or* $\mathrm{CP}^2$ *or the universal covering* $\tilde{M}$ *of M is isometric to* $S^2 \times S^2$, *with their normalized metrics. If* $|W^+|^2 \geq 2/3$, *then M is isometric to* $\mathrm{CP}^2$ *or* $\tilde{M}$ *is isometric to* $S^2 \times S^2$, *with their normalized metrics*.
(d) *Let M be oriented definite with signature* $\tau \neq 0$. *If* $\inf K \geq (2-\sqrt{5})\rho/6$ *or* $\sup K \leq (2+\sqrt{5})\rho/6$, *then M is isometric to* $\mathrm{CP}^2$.

**Corollary 1.1.** *Let M be a compact-oriented Einstein four-manifold with sectional curvature* $K \geq 0$. *If* $|W|$ *is constant, then M is isometric to either* $S^4$ *or* $\mathrm{CP}^2$ *or* $\tilde{M}$ *is isometric to* $S^2 \times S^2$.

An Einstein manifold $M$ with metric $g$ is rigid (see Definition 12.64 of [1]), if in a small neighborhood of $g$ there is no other Einstein metric. According to Bourguignon (see Corollary 12.72 of [1]), if $M$ has dimension 4 and its sectional curvature $K$ satisfies $K_0/6 < K \leq K_0$, then $M$ is rigid. Note that in this case, $\sup K < 3\rho/4$, where $\rho$ is the Ricci curvature of $M$. This suggests that Theorem 1.2(a) can be improved for $\sup K \leq 3\rho/4$. In fact we have the following proposition.

**Proposition 1.1.** *A compact Einstein four-manifold with Ricci curvature* $\rho > 0$ *is rigid, if the sectional curvature K satisfies* $\sup K < 3\rho/4$.

Another question can be considered: A Riemannian $n$-manifold $M$ has pure *curvature operator*, if for each point $x$ of $M$, there exists a basis $\{X_1, X_2, \dots, X_n\}$ of the tangent space $T_xM$ such that all exterior products $X_i \wedge X_j$ $(i < j)$ are eigenvectors of the curvature operator of $M$. For example, $S^4$ and $S^2 \times S^2$ have pure curvature operator but $\mathrm{CP}^2$ does not have this property. Also, an Einstein four-submanifold of the Euclidean space $R^6$ with positive Ricci curvature has pure curvature operator (see Proposition 3.1). For these manifolds, we can improve Theorem 1.2(a).

**Theorem 1.3.** *Let M be a compact Einstein four-manifold with Ricci curvature* $\rho > 0$ *and sectional curvature K such that* $\sup K \leq \rho$:

(a) *If M has pure curvature operator, then M is isometric to either* $S^4$, $\mathrm{RP}^4$ *or the universal covering* $\tilde{M}$ *of M is isometric to* $S^2 \times S^2$.
(b) *If M is a submanifold of the Euclidean space* $R^6$, *then M is isometric to either* $S^4$ or $S^2 \times S^2$.

## 2. Preliminaries

For an oriented four-dimensional manifold $M$ the bundle of two-forms splits $\wedge^2 M = \wedge^+ \otimes \wedge^-$ into $+1$-eigenspace of the Hodge*-operator and $-1$-eigenspace. The Weyl curvature tensor $W$ is an endomorphism of $\wedge^2 M$ such that $W = W^+ \otimes W^-$, where $W^\pm : \wedge^\pm \to \wedge^\pm$. Note that $W^\pm$ can be viewed as (0, 4)-tensor and $W^\pm|_{\wedge^\mp} \equiv 0$. If $M$ is an Einstein four-manifold with Ricci curvature $\rho$ and let $x \in M$, we use the normal form for the curvature operator of $M$ (see [2,8]). Then there exists an orthonormal basis for $\wedge_x^2$ such that the curvature operator takes the form

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

where $A$ and $B$ are given, respectively, by

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}.$$

In particular, there exists an orthonormal basis of $\wedge_x^\pm$ for $W^\pm(x)$ with respective eigenvalues $\lambda_i \pm \mu_i - \rho/3$, for $i = 1, 2, 3$. Moreover, the $\lambda_i$'s are relative critical values of sectional curvature and

$$\sum_{i=1}^{3} \lambda_i = \rho. \tag{2.1}$$

Also, it follows of the first Bianchi's identity that

$$\sum_{i=1}^{3} \mu_i = 0. \tag{2.2}$$

By Lemma 2 of Berger [2],

$$|\mu_i - \mu_j| \le \lambda_i - \lambda_j, \tag{2.3}$$

where $i > j$ and $\lambda_1 \le \lambda_2 \le \lambda_3$. Using this basis, (1.1) and (1.2) become

$$\chi = \frac{1}{4\pi^2} \int_M \left[ \sum_{i=1}^{3} (\lambda_i^2 + \mu_i^2) \right] \mathrm{d}V \tag{2.4}$$

and

$$\tau = \frac{1}{3\pi^2} \int_M \left[ \sum_{i=1}^{3} \lambda_i \mu_i \right] \mathrm{d}V. \tag{2.5}$$

We have the following lemma.

**Lemma 2.1.** *Let $M$ be a compact-oriented Einstein four-manifold with Ricci curvature $\rho$, volume $V$, sectional curvature $K$, Euler characteristic $\chi$ and signature $\tau$. Let $K_0 = \inf K$ or $K_0 = \sup K$.*

(a) *Then*

$$\chi \leq \frac{(5\rho^2 + 36K_0^2 - 24\rho K_0)V}{12\pi^2}.$$

*Moreover, if we have the equality above and* $\rho > 0$, *then M is isometric to either* $S^4$ *or* $\mathrm{CP}^2$.

(b) *If M has* $W^{\pm} \not\equiv 0$ *and* $\rho > 0$, *then*

$$\chi > \frac{3(36\alpha^2 - 24\alpha + 5)}{2(36\alpha^2 - 24\alpha + 2)}|\tau|,$$

*where* $\alpha = K_0/\rho$.

**Corollary 2.1.** *Let M be a compact-oriented Einstein four-manifold with Ricci curvature* $\rho > 0$, *sectional curvature K, Euler characteristic* $\chi$ *and signature* $\tau$. *Let* $\alpha = \sup K/\rho$ *or* $\alpha = \inf K/\rho$. *If n is the number of elements of the fundamental group of M, then*

(a) *M is isometric to* $S^4$ *or*

$$n < \frac{2(36\alpha^2 - 24\alpha + 5)}{\chi}.$$

(b) *If* $\inf K \geq (2 - \sqrt{\chi - 1})\rho/6$ *or* $\sup K \leq (2 + \sqrt{\chi - 1})\rho/6$ *then M is simply connected.*

**Remark 2.1.** In [8, Remark 5], Hitchin proposed the problem to know if the existence of an Einstein metric implies anything about the fundamental group of a four-manifold. Corollary 2.1 is a partial response of this question.

**Proof of Lemma 2.1.**

(a) Let $\rho$ be the Ricci curvature of $M$ and $K$ its sectional curvature. In accordance with (2.3), $\mu_i^2 + \mu_j^2 - 2\mu_i\mu_j \leq \lambda_i^2 + \lambda_j^2 - 2\lambda_i\lambda_j$, for $i > j$. Using (2.1) and (2.2) we have

$$6\sum_{i=1}^{3}(\lambda_i^2 + \mu_i^2) \leq 10\sum_{i=1}^{3}\lambda_i^2 - 4\sum_{i>j}\lambda_i\lambda_j = 10\rho^2 - 24\sum_{i>j}\lambda_i\lambda_j. \tag{2.6}$$

Let $K_0 = \inf K$ or $K_0 = \sup K$. Then

$$(\lambda_i - K_0)(\lambda_j - K_0) \geq 0. \tag{2.7}$$

By (2.6) and (2.7) we have $\sum_{i=1}^{3}(\lambda_i^2 + \mu_i^2) \leq 5\rho^2/3 + 36K_0^2 - 24K_0\rho$ and so, it follows from (2.4) that

$$\chi \leq \frac{(5\rho^2 + 36K_0^2 - 24K_0\rho)V}{12\pi^2}. \tag{2.8}$$

If we have an equality in (2.8), then (2.7), (2.6) and (2.3) become equalities. In particular, from the equality in (2.7) we have $\lambda_1 = \lambda_2 = K_0$, $\lambda_3 = \rho - 2K_0$ if $K_0 = \inf K$ and $\lambda_2 = \lambda_3 = K_0$, $\lambda_1 = \rho - 2K_0$ if $K_0 = \sup K$. In both cases, $|W^{\pm}|$ and $\det W^{\pm}$ are constants and then by Proposition 9 of [5], $M$ is locally symmetric and moreover the universal covering $\tilde{M}$ of $M$ is symmetric. By Jensen [9], $\tilde{M}$ is isometric to either $S^2 \times S^2$, $S^4$ or $\mathrm{CP}^2$. Let $\rho > 0$ and assume that $\tilde{M}$ is isometric to $S^2 \times S^2$. Then $\sup K =$ curvature of $S^2 = \rho$ and $\inf K = 0$. If $K_0 = \sup K$ then $\lambda_1 = \rho - 2K_0 = -\rho < 0$ (contradiction). If $K_0 = \inf K = 0$, by equality in (2.8) we have that $\chi = 5\rho^2 V/12\pi^2$. On the other hand, since $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = \rho$, it follows from equalities in (2.3) and (2.2) that $2\mu_1 = 2\mu_2 = -\mu_3$ and $|3\mu_3| = \rho$. In accordance with (2.4), $\chi = 7\rho^2 V/24\pi^2$ (contradiction). So $\tilde{M}$ is isometric to either $S^4$ or $\mathrm{CP}^2$. In particular, $M$ has positive sectional curvature and by Synge's lemma $M = \tilde{M}$. This proves Lemma 2.1(a).

(b) Let $M$ be a compact-oriented Einstein four-manifold with Ricci curvature $\rho > 0$ and $W^{\pm} \not\equiv 0$. By Theorem 1 and Corollary 1 of [7],

$$\int_M |W^{\pm}|^2 \, \mathrm{d}V \geq \frac{2\rho^2 V}{3}.$$

Using (1.1) and (1.2),

$$\chi \geq \frac{3|\tau|}{2} + \frac{\rho^2 V}{4\pi^2}.$$

On the other hand, it follows from the result of Hitchin [1, Theorem 13.30] that $M$ is non-isometric to either $S^4$ or $\mathrm{CP}^2$ and so, using the stricty inequality in Lemma 2.1(a), we obtain that

$$\chi > \frac{3}{2}\left(\frac{36\alpha^2 - 24\alpha + 5}{36\alpha^2 - 24\alpha + 2}\right)|\tau|,$$

where $\alpha = \inf K/\rho$ or $\alpha = \sup K/\rho$. □

**Proof of Corollary 2.1.**

(a) Let $M$ be a compact-oriented Einstein four-manifold with Ricci curvature $\rho > 0$. If $M$ is non-isometric to $S^4$, then the universal covering $\tilde{M}$ of $M$ is non-isometric to $S^4$ and has Ricci curvature $\rho > 0$. Bishop's inequality then asserts that $\tilde{M}$ has volume $\tilde{V} < 2\pi^2/\rho^2$. If $n$ is the number of elements of the fundamental group of $M$, then the Euler characteristic $\tilde{\chi}$ of $\tilde{M}$ satisfies $\tilde{\chi} = n\chi$, where $\chi$ is the Euler characteristic of $M$. Using Lemma 2.1(a) for $\tilde{M}$, we have that $n\chi < 2(36K_0^2 - 24\rho K_0 + 5\rho^2)/\rho^2$ and this proves Corollary 2.1(a).

(b) Let $\rho > 0$ the Ricci curvature of $M$ and $K$ its sectional curvature. Note that if $\inf K \geq (2 - \sqrt{\chi - 1}\rho)/6$ or $\sup K \leq (2 + \sqrt{\chi - 1}\rho)/6$, then

$$\frac{2(36\alpha^2 - 24\alpha + 5)}{\chi} \leq 2,$$

where $\alpha = \inf K/\rho$ or $\alpha = \sup K/\rho$. So, by Corollary 2.1(a), $n = 1$ and this proves that $M$ is simply connected. □

## 3. Proofs

**Proof of Theorem 1.1.** Let $M$ be a compact-oriented Einstein four-manifold with Ricci curvature $\rho$, volume $V$ and sectional curvature $K$. If $\rho < 0$ and $K_0 = \inf K \geq 2\rho/3$ then $K_0 = \inf K \leq 0$. So, $36K_0^2 - 24K_0\rho \leq 0$ and it follows from Lemma 2.1 that

$$\chi \leq \frac{5\rho^2 V}{12\pi^2}. \tag{3.1}$$

On the other hand, using (2.4) and (2.5),

$$2\chi \pm 3\tau = \frac{1}{2\pi^2}\int_M \sum_{i=1}^{3}[\lambda_i \pm \mu_i]^2 \, dV$$

and so, by the Schwarz inequality,

$$2\chi - 3|\tau| \geq \frac{\rho^2 V}{6\pi^2}.$$

Using (3.1), we have that $\chi \geq (15/8)|\tau|$. □

**Proof of Theorem 1.2.**

(a) Let $\rho > 0$ and $\sup K \leq 2\rho/3$. If $M$ is non-orientable, consider the two-fold covering of $M$. From (2.3), $\lambda_1 \pm \mu_1 \leq \lambda_2 \pm \mu_2 \leq \lambda_3 \pm \mu_3$.

Using (2.1) and (2.3), $\lambda_3 \pm \mu_3 \leq 2\lambda_3 - \rho/3 \leq \rho$. So, the operators $P_\pm = (2\rho/3)I - W^\pm$, where $I$ is the identity of $\wedge^\pm$, are non-negative operators. In accordance with the proof of Theorem 4.2 of [11], $\nabla W^\pm \equiv 0$. If $X$ is a vector field on $M$, then $X(|W^\pm|^2) = 2 < \nabla_X^{W^\pm}, W^\pm > \equiv 0$ and so $|W^\pm|$ are constants. By the Weitzenbock formulas

$$\Delta|W^\pm|^2 = 2|\nabla W^\pm|^2 + 4\rho|W^\pm|^2 - 36 \det W^\pm,$$

the determinants $\det W^\pm$ are constants. So, it follows from Proposition 9 of [5] that $M$ is locally symmetric and the universal covering $\tilde{M}$ of $M$ is symmetric. Second Jensen [9], $M$ is isometric to either $S^4$, $\mathrm{CP}^2$ or $S^2 \times S^2$. On the other hand, since $S^2 \times S^2$ has $\sup K = \rho$, then $M$ is isometric to either $S^4$ or $\mathrm{CP}^2$. If $M$ is not orientable then $M$ is isometric to $\mathrm{RP}^4$, since the only locally symmetric quotient of $S^4$ or $\mathrm{CP}^2$ is $\mathrm{RP}^4$.

(b) Let $M$ be a compact Einstein four-dimensional manifold with Ricci curvature $\rho = 1$, sectional curvature $K$ and volume $V$. If $M$ is non-orientable, consider the two-fold covering of $M$ and assume that $M$ is non-isometric to either $S^4$ or $\mathrm{CP}^2$. From Hitchin (see [1, Theorem 13.30]), $W^\pm \not\equiv 0$. In this case, it follows from Theorem 1 and Corollary 1 of [7] that

$$\int_M |W^\pm|^2 \, dV \geq \frac{2\rho^2 V}{3}. \tag{3.2}$$

So, in accordance with (1.1) and (3.2), we have

$$\chi \geq \frac{\rho^2 V}{4\pi^2}. \tag{3.3}$$

On the other hand, from Lemma 2.1 and our hypothesis on $M$,

$$\chi < [5\rho^2 + 36K_0^2 - 24K_0\rho]\frac{V}{12\pi^2}, \tag{3.4}$$

where $\rho = 1$ and $K_0 = \inf K$. Note that $K_0 \leq 1/3$. By using (3.3) and (3.4), we have that $K_0 < (2-\sqrt{2})/6$ or $K_0 > (2+\sqrt{2})/6$. By our hypothesis, $K_0 \geq (2-\sqrt{2})/6$. Then $K_0 > (2+\sqrt{2})/6 > 1/3$ (contradiction). This proves that $M$ is isometric to either $S^4$ or $\mathrm{CP}^2$, with their normalized metrics. If $M$ is not orientable then $M$ is isometric to $\mathrm{RP}^4$, since the only locally symmetric quotient of $S^4$ or $\mathrm{CP}^2$ is $\mathrm{RP}^4$. This proves Theorem 1.2(b).

(c) Let $\rho = 1$, $K \geq 0$ and $M$ oriented. If $W^+ \equiv 0$ or $W^- \equiv 0$, then it follows Hitchin (see [1, Theorem 13.30]) that $M$ is isometric to either $S^4$ or $\mathrm{CP}^2$. Let $W^\pm \not\equiv 0$. If $|W^+|^2 \leq 2/3$, it follows from (3.2) that $|W^+|^2 = 2/3$. But, from Lemma 1 of [7], $|W^-|^2 \leq 2/3$. So, by (3.2) we have that also $|W^-|^2 = 2/3$. Moreover, by Theorem 1 and Corollary 1 of [7], $\nabla W^\pm \equiv 0$. Using the same arguments of the proof of (a), we have that $\tilde{M}$ is isometric to $S^2 \times S^2$, since $W^\pm \not\equiv 0$. Let $|W^+|^2 \geq 2/3$. From Lemma 1 of [7], $|W^-|^2 \leq 2/3$. In accordance with (3.2), $|W^-|^2 = 2/3$. Also, by Lemma 1 of [7], $|W^+|^2 \leq 2/3$ and so $|W^+|^2 = 2/3$. As in the previous case, we have that $\tilde{M}$ is isometric to $S^2 \times S^2$.

(d) Let $M$ be a compact-oriented definite Einstein four-manifold with Ricci curvature $\rho > 0$ and signature $\tau \neq 0$. Then $M$ is non-isometric to $S^4$. Assume that $M$ is non-isometric to $\mathrm{CP}^2$. So, $W^\pm \not\equiv 0$. On the other hand, since $M$ is definite, then $\chi = 2 + |\tau|$. Using Lemma 2.1(b), we have that

$$|\tau| < \frac{4(36\alpha^2 - 24\alpha + 2)}{(36\alpha^2 - 24\alpha + 11)} \leq 1,$$

since $\alpha = \inf K/\rho \geq (2-\sqrt{5})/6$ or $\alpha = \sup K/\rho \leq (2+\sqrt{5})/6$. But, this contradicts the fact of that $\tau \neq 0$. So, $M$ is isometric to $\mathrm{CP}^2$. □

**Proof of Corollary 1.1.** Assume that $M$ has Ricci curvature $\rho = 1$. So, Corollary 1.1 is consequence of Theorem 1.2(c). □

**Proof of Proposition 1.1.** In accordance with Lemma 12.71 and Theorem 12.67 of [1], $M$ is rigid if

$$\min 2\sup K - \rho, \rho - 4\inf K < \tfrac{1}{2}\rho.$$

Since $M$ is compact, there exists $x \in M$ and there exists an orthonormal basis $\{X_1, X_2, X_3, X_4\}$ of $T_xM$ such that $\sup K = K(X_1, X_2) = \rho - K(X_1, X_3) - K(X_1, X_4)$. So,

$$2\sup K - \rho \leq \rho - 4\inf K.$$

Moreover, $2\sup K - \rho < \rho/2$, since $\sup K < 3\rho/4$. So, $M$ is rigid.

The following lemma characterizes the Einstein four-manifolds with pure curvature operator. □

**Lemma 3.1.** *Let $M$ be an oriented Einstein four-manifold. Then $M$ has pure curvature operator if and only if* $|W^+| = |W^-|$ *and* $\det W^+ = \det W^-$. *Moreover,*

$$\chi \leq \frac{(\rho^2 + 6K_0^2 - 4\rho K_0)V}{4\pi^2}.$$

*where* $K_0 = \inf K$ *or* $K_0 = \sup K$.

**Proof of Lemma 3.1.** If $M$ has pure curvature operator, the result follows from Lemma 16.20 of [1]. Let $|W^+| = |W^-|$ and $\det W^+ = \det W^-$. Let $x \in M$ and $R$ the tensor of the curvature of $M$. In accordance with Berger [2, Lemma 2], there exists an orthonormal basis $\{X_1, X_2, X_3, X_4\}$ of the tangent space $T_xM$ such that if $R_{ijkl} = \langle R(X_i, X_j)X_k, X_l\rangle$ $(i, j, k, l = 1, 2, 3, 4)$, then $R_{ijkl} = 0$ for all $i \neq j$. Moreover, $R_{1234} = \mu_3$, $R_{1342} = \mu_2$, $R_{1423} = \mu_1$, where the $\mu_i$ satisfies (2.2). On the other hand, note that $M$ has pure curvature operator (in $x$) if $R_{ijkl} = 0$ and the set of the elements $\{i, j, k, l\}$ contains more than two elements. So, by the Bianchi identities, it is sufficient to prove that $\mu_i = 0$ for $i = 1, 2, 3$. For this, let $\alpha_i = \lambda_i - \rho/3$, where $\lambda_i$ satisfies (2.1). Using $|W^+| = |W^-|$ and (2.2) we have

$$\sum_{i=1}^{3} \alpha_i = \sum_{i=1}^{3} \alpha_i\mu_i = 0. \tag{3.5}$$

Moreover, since $\det W^+ = \det W^-$,

$$(\alpha_1 - \mu_1)(\alpha_2 - \mu_2)(\alpha_3 - \mu_3) = (\alpha_1 + \mu_1)(\alpha_2 + \mu_2)(\alpha_3 + \mu_3). \tag{3.6}$$

Then, it follows from (3.6), (3.5) and (2.1) and (2.2) that

$$\begin{aligned}(2\alpha_3^2 + \alpha_1\alpha_2 + \mu_1\mu_2)\mu_3 &= (2\alpha_1^2 + \alpha_2\alpha_3 + \mu_2\mu_3)\mu_1 \\ &= (2\alpha_2^2 + \alpha_1\alpha_3 + \mu_1\mu_3)\mu_2 = 0.\end{aligned} \tag{3.7}$$

Suppose that $\mu_1 = 0$. Using (3.5), (2.1)–(2.3) and (3.7) we have that $\mu_1 = \mu_2 = \mu_3 = 0$. The cases $\mu_2 = 0$ or $\mu_3 = 0$ are similar. Let $\mu_i \neq 0$ for all $i = 1, 2, 3$. In accordance with (3.7),

$$\begin{aligned}2\alpha_3^2 + \alpha_1\alpha_2 + \mu_1\mu_2 &= 2\alpha_1^2 + \alpha_1^2 + \alpha_2\alpha_3 + \mu_2\mu_3 \\ &= 2\alpha_2^2 + \alpha_1\alpha_3 + \alpha_1\alpha_3 + \mu_1\mu_3 = 0.\end{aligned} \tag{3.8}$$

So, using (3.8), (3.5) and (2.2), we have that $3\sum_{i=1}^{3} \alpha_i^2 = \sum_{i=1}^{3} \mu_i^2$ or

$$3\sum_{i=1}^{3} \lambda_i^2 - \rho^2 = \sum_{i=1}^{3} \mu_i^2. \tag{3.9}$$

Using (2.6),

$$\sum_{i=1}^{3} \mu_i^2 \leq \sum_{i=1}^{3} \lambda_i^2 - \frac{\rho^2}{3} \leq \frac{1}{3}\sum_{i=1}^{3} \mu_i^2.$$

So, $\mu_i = 0$ for $i = 1, 2, 3$ (contradiction). This proves that $M$ has pure operator curvature. In particular, (2.4) becomes

$$\chi = \frac{1}{4\pi^2} \int_M \sum_{i=1}^{3} \lambda_i^2 \, \mathrm{d}V.$$

Note that $\sum_{i=1}^{3} \lambda_i^2 = \rho^2 - 2\sum_{i>j} \lambda_i \lambda_j$.

If $K_0 = \inf K$ or $K_0 = \sup K$, then $(\lambda_i - K_0)(\lambda_k - K_0) \geq 0$ and $\sum_{i=1}^{3} \lambda_i^2 \leq \rho^2 + 6K_0^2 - 4\rho K_0$. So

$$\chi \leq \frac{(\rho^2 + 6K_0^2 - 4\rho K_0)V}{4\pi^2}. \qquad \square$$

**Proof of Theorem 1.3.**

(a) Let $\rho > 0$ be the Ricci curvature of $M$, where $M$ is a compact Einstein four-manifold with pure operator curvature. By the proof of Lemma 3.1 we have that $\mu_i = 0$ for $i = 1, 2, 3$. Then the operators $(2\rho/3)I - W^{\pm}$ has eigenvalues $\rho - \lambda_i \geq 0$, where the $\lambda_i$ satisfies (2.1). In this case, the proof of Theorem 1.3(a) is similar to the proof of Theorem 1.2(a) and we can deduce that $M$ is isometric to either $S^4$ or $\mathrm{RP}^4$ or $\tilde{M}$ is isometric to $S^2 \times S^2$, since that $\mathrm{CP}^2$ does not have pure curvature operator.

(b) Let $M$ be a submanifold of $R^6$, where $M$ is a compact Einstein four-manifold with Ricci curvature $\rho > 0$. Assumes initially that $M$ has pure curvature operator. By Theorem 1.3(a), $M$ has non-negative sectional curvature. In particular, since that $M$ has finite fundamental group, it follows from the result of [4] that $M$ is orientable. Moreover, by Theorem 2.2 of [3], $M$ is simply connected. So, $M$ is isometric to either $S^4$ or $S^2 \times S^2$. The proof of the fact that $M$ has pure operator curvature follows from the following more general result: $\square$

**Proposition 3.1.** *Let $f : M^n \to Q_c^{n+2}$, be an isometric immersion, where $n \geq 3$, $M^n$ is an Einstein n-manifold with Ricci curvature $\rho$ and $Q_c^{n+2}$ is a space of constant sectional curvature c. If $\rho > (n-1)c$, then f has flat normal connection. In particular, M has pure curvature operator.*

**Proof of Proposition 3.1.** Let $x \in M$, $\{\xi_1, \xi_2\}$ an orthonormal set in $(T_xM)^{\perp}$ and let $A_1 = A_{\xi_1}$, $A_2 = A_{\xi_2}$ the Weingarten operators in directions $\xi_1$ and $\xi_2$, respectively. It follows from the Gauss equation that

$$A_1^2X - (\mathrm{tr}\, A_1)A_1X + A_2^2X - (\mathrm{tr}\, A_2)A_2X = \alpha X, \tag{3.10}$$

where $X \in T_xM$ and $\alpha = -\rho + (n-1)c$. Let $H = (1/n)(\mathrm{tr}\, A_1^2 + \mathrm{tr}\, A_2^2)^{1/2}$ be the mean curvature of immersion. Note that $H > 0$, since $\alpha < 0$. We can assume that $\xi_1 = (1/H)\vec{H}$, where $\vec{H}$ is the mean curvature vector of immersion. Then $\mathrm{tr}\, A_1 = nH$, $\mathrm{tr}\, A_2 = 0$ and (3.10) becomes

$$A_1^2X - nHA_1X + A_2^2X = \alpha X. \tag{3.11}$$

Note that if $\lambda$ is an eigenvalue of $A_1$, then it follows from (3.11) that $\lambda > 0$. Moreover, (3.11) is equivalent to

$$A^2 + B^2 = \beta I, \tag{3.12}$$

where $A = (A_1 - (nH/2)I)^2$, $B = A_2$, $I$, is the identity of $T_xM$ and $\beta = (n^2H^2/4) + \alpha$. The immersion has flat normal connection if the Weingarten operators $A_1$ and $A_2$ are commutative. Then, it is sufficient to prove that $AB = BA$. Note that $A$ does not admit eigenvalues of this form $\pm\mu$, with $\mu \neq 0$. Also, we have two cases: $\beta = 0$ or $\beta > 0$. If $\beta = 0$, it follows from (3.12) that $A = B = 0$ and $AB = BA$. Let $\beta > 0$ and let $\{X_1, \ldots, X_n\}$ be an orthonormal basis of $T_xM$ such that $AX_i = \mu_i X_i$ $(i = 1, 2, \ldots, n)$. Let $X = X_1$. Then $BAX = \mu_1 BX$. If $\mu_1 = 0$ then $BAX = 0$ and using (3.12), we have

$$|ABX|^2 = \langle ABX, ABX\rangle = \langle A^2BX, BX\rangle = \langle BA^2X, BX\rangle = 0.$$

Then, in this case $ABX = BAX = 0$. Let $\mu_1 \neq 0$. Then

$$\begin{aligned}\langle BAX - ABX, X_i\rangle &= \langle BAX, X_i\rangle - \langle ABX, X_i\rangle \\ &= \mu_1\langle BX, X_i\rangle - \langle BX, AX_i\rangle = (\mu_1 - \mu_i)\langle BX, X_i\rangle.\end{aligned} \tag{3.13}$$

If $\mu_1 = \mu_i$ then $\langle BAX - ABX, X_i\rangle = 0$. Let $\mu_1 \neq \mu_i$. By hypothesis, $\mu_1 \neq -\mu_i$. Using (3.12), we have $X_i = (1/\beta)(B^2X_i + \mu_i^2 X_i)$ and $B^2X = (\beta - \mu_1^2)X$. Then, it follows from (3.13) that

$$(\mu_1 - \mu_i)\langle BX, X_i\rangle = \frac{\mu_1 - \mu_i}{\beta}[\mu_i^2 + \beta - \mu_1^2]\langle BX, X_i\rangle.$$

So, $\langle BX, X_i\rangle = 0$ and follows from (3.13), that $\langle BAX - ABX, X_i\rangle = 0$. This proves that $BA = AB$ and the immersion has flat normal connection. In particular, $M$ has pure operator curvature. □

**Remark 3.1.** Proposition 3.1 generalizes Theorem 2 of Erbacher [6], who proved a similar result, when $M^n$ is another space with constant sectional curvature.

## Acknowledgements

The author thanks the referee for his suggestions and corrections.

## References

[1] A. Besse, Einstein Manifolds, Springer-Verlag, 1980.
[2] M. Berger, Sur quelques varietes d'Einstein compacts, Ann. Mat. Pur. Appl. 53 (1961) 89–96.
[3] Y. Baldin, F. Mercuri, Isometric immersions in codimension two with nonnegative curvature, Math. Z. 173 (1980) 111–117.
[4] Y. Baldin, F. Mercuri, Codimension two nonorientable submanifolds with nonnegative curvature, Proc. AMS 103 (3) (1988) 18–20.

[5] A. Derdzinsky, Self-dual Kaehler manifolds and Einstein manifolds of dimension four, Comp. Math. 49 (1983) 405–433.
[6] J. Erbacher, Isometric immersions of constant mean curvature and triavility of the normal connection, Nagoya Math. J. 45 (1971) 139–165.
[7] M.J. Gursky, C. LeBrun, On Einstein manifolds of positive sectional curvature, Ann. Global Anal. Geom. 17 (1999) 315–328.
[8] N.J. Hitchin, On compact four-dimensional Einstein manifolds, J. Diff. Geom. 9 (1974) 435–442.
[9] G. Jensen, Homogeneous Einstein spaces of dimension four, J. Diff. Geom. 3 (1969) 309–349.
[10] C. LeBrun, M. Wang, Essays on Einstein manifolds. Surveys in Differential Geometry, vol. VI, International Press, Boston, MA, 1999.
[11] M.J. Micallef, M.Y. Wang, Metrics with nonnegative isotropic curvature, Duke Math. J. 72 (3) (1993) 649–672.
[12] D.G. Yang, Rigidity of Einstein 4-manifolds with positive curvature, Invent. Math. 142 (2000) 435–450.